Transmission of Sound through Annular Ducts of Varying Cross Sections

A. H. Nayfeh,* J. E. Kaiser,† and D. P. Telionis‡
Virginia Polytechnic Institute and State University, Blacksburg, Va.

An asymptotic solution is presented for the transmission and attenuation of acoustic waves in an annular duct of slowly-varying cross section which carries a sheared mean flow. The analysis also takes into account the growth of the boundary layer as well as any slow variations in the acoustic liner properties. The problem is reduced to the solution of a first-order ordinary differential equation for the amplitude. The analysis is used to estimate the effects of the variations of the duct cross section and boundary layer growth on the different acoustic modes.

Introduction

THE prediction of wave propagation in ducts of variable cross sections is a problem whose solution has application to the design of numerous facilities, such as central air conditioning and heating installation, loud speakers, high-speed wind tunnels, aircraft engine-duct systems, and rocket nozzles. Concern about the environmental impact of modern technology has focused attention on the problem of noise alleviation and thus increased the importance of obtaining accurate predictions of the characteristics of sound propagation in realistic duct configurations. In many instances such ducts carry a high-speed mean flow that possesses strong transverse velocity and temperature gradients, and the sound generated within the duct is of a complexity that requires the analysis of numerous modes in order to accurately predict the sound characteristics. Thus, there is need for a method that can handle the effects of a general mean flow—including both streamwise and transverse gradients—and is not restricted to the analysis of the plane modes of sound propagation.

Early studies of wave propagation through variable-area ducts considered the case of no mean flow. Webster's equation[1] for the propagation of a plane wave is now a classic and quite a few investigators verified or expanded this theory (for more references see the review article, Ref. 2). Isakovitch,[3] Salant,[4] and Nayfeh[5] studied the wave propagation through ducts with weak sinusoidal nonuniformities. Several authors[6-8] utilized a discretization technique based on the solution for a single duct discontinuity. Nayfeh and Telionis[9] employed the method of multiple scales and derived a perturbation solution for a duct having an arbitrary but slowly-varying cross section with axial distance. Beckemeyer and Eversman[10] used the Ritz minimization of functionals with the governing equations as stationary conditions in order to waive the restriction of slow variation with the axial distance; however, it is not known whether their analysis will converge when the duct variations are large.

The inclusion of the effects of a mean flow makes the problem more difficult. Most studies of this problem have employed one or more simplifying assumptions. The most commonly-used assumption is that of quasi-one-dimensional flow, which eliminates the effect of sound refraction through the boundary layer. Powell[11] studied the propagation of sound discontinuities, while Eisenberg and Kao[12] considered the propagation of the lowest acoustic mode through ducts with variable cross sections. Huerre and Karamcheti[13] and King and Karamcheti[14] investigated the propagation of the lowest acoustic mode using, respectively, the short-wave approximation (ray acoustics) and the method of characteristics. Hogge and Ritzi[15] assumed that the duct can be broken down into cylindrical and conical sections, and they solved the wave equation for a uniform mean flow by application of the method of finite elements.

In studies of wave propagation in constant area ducts, it has been shown that higher acoustic modes play an important role[16] in the sound-propagation characteristics and that gradients in the mean flow have a strong refracting effect (see, for example, Ref. 2). Hence, for a study of propagation in variable-area ducts, a method that is not restricted to a study of the fundamental mode or to a study of quasi-one-dimensional mean flows is needed. Perhaps the only simplifying factor of the problem is the fact that, in many practical situations, for example in the bypass ducts of high-bypass-ratio jet engines, the duct cross-sectional area depends only mildly on the distance along the axis of the duct. Moreover, the growth of the mean boundary layer at high Reynolds numbers is a slow function of the axial distance, except perhaps at the leading edge of the duct inlet. The above factors indicate that a perturbation method should be appropriate. Such a method could determine the correction due to weak nonuniformities of the duct shape, a mean flow with a small normal velocity component, slow variations of the duct liner properties, and a growing boundary-layer thickness, in addition to including the effects of large transverse velocity and temperature gradients.

By using the Born approximation Tam[17] investigated the transmission and scattering of spinning acoustic-wave modes through a duct nonuniformity. With this perturbation solution, he introduced for the first time the effect of the normal component of the mean flow but neglected the refractive effect of transverse velocity gradients. Nayfeh, Telionis, and Lekoudis[18] used the method of multiple scales in order to study the propagation of all acoustic modes in a two-dimensional duct with a slowly-varying cross section carrying a sheared mean flow. The purpose of the present paper is to extend the work of Ref. 18 to an axisymmetric configuration. The duct is assumed to be annular, permitting application of the analysis to bypass ducts of a jet engine. The analysis includes the effects of slow axial variations of the cross sectional area, of the liner properties and of the boundary-layer thickness, the effect of the normal mean velocity and the effect of transverse gradients in the mean velocity, but does not include compressibility effects.

Presented as Paper 74-58 at the AIAA 12th Aerospace Sciences Meeting, Washington, D.C., January 30–February 1, 1974; submitted February 21, 1974; revision received July 19, 1974. This work was supported by the Loads Division of the NASA Langley Research Center Under Grant NGR 47-004-109.

Index category: Aircraft Noise; Powerplant.

* Professor, Engineering Science and Mechanics Department. Member AIAA.

† Assistant Professor, Engineering Science and Mechanics Department. Member AIAA.

‡ Associate Professor, Engineering Science and Mechanics Department. Member AIAA.

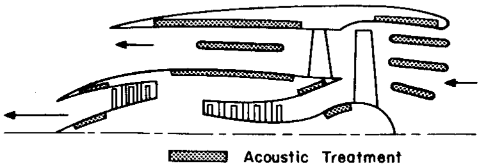

Fig. 1 Schematic of a high bypass jet engine.

Problem Formulation

We consider the transmission and attenuation of acoustic waves in an annular duct (Figs. 1 and 2) having a slowly varying cross section and carrying a sheared mean flow. We assume the mean flow to have constant density ρ_0 and pressure p_0. Moreover, we consider the mean flow to be slightly nonparallel; that is, the mean-velocity components along and normal to the duct are assumed to have the form $U(x_1, r)$ and $\varepsilon V(x_1, r)$, where $x_1 = \varepsilon x$ is a slow scale and ε is a small dimensionless parameter representing, for example, the maximum slope of the wall.

We assume that each flow quantity is the sum of a mean part and an acoustic part; that is, we express the flow quantities as

$$U(x_1,r)+\varepsilon_1 u(x,r,\theta,t), \quad \varepsilon V(x_1,r)+\varepsilon_1 v(x,r,\theta,t)$$
$$\varepsilon_1 w(x,r,\theta,t), \quad p_0+\varepsilon_1 p(x,r,\theta,t), \quad \rho_0+\varepsilon_1\rho(x,r,\theta,t) \tag{1}$$

where ε_1 is a small dimensionless parameter characterizing the amplitude of the acoustic wave. Substituting Eq. (1) into the Euler equations and subtracting the mean-flow quantities, we obtain

$$\frac{\partial\rho}{\partial t}+U\frac{\partial\rho}{\partial x}+\rho_0\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial w}{\partial\theta}+\frac{v}{r}\right)= -\varepsilon V\frac{\partial\rho}{\partial r}+O(\varepsilon_1)+O(\varepsilon\varepsilon_1) \tag{2}$$

$$\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}+V\frac{\partial U}{\partial r}+\frac{1}{\rho_0}\frac{\partial p}{\partial x}= -\varepsilon\left[V\frac{\partial u}{\partial r}+u\frac{\partial U}{\partial x_1}+\frac{\rho}{\rho_0}\left(U\frac{\partial U}{\partial x_1}+V\frac{\partial U}{\partial r}\right)\right]+O(\varepsilon_1)+O(\varepsilon\varepsilon_1) \tag{3}$$

$$\frac{\partial v}{\partial t}+U\frac{\partial v}{\partial x}+\frac{1}{\rho_0}\frac{\partial p}{\partial r}=-\varepsilon\frac{\partial}{\partial r}(vV)+O(\varepsilon^2)+O(\varepsilon_1)+O(\varepsilon\varepsilon_1) \tag{4}$$

$$\frac{\partial w}{\partial t}+U\frac{\partial w}{\partial x}+\frac{1}{\rho_0 r}\frac{\partial p}{\partial\theta}=-\varepsilon\left(\frac{Vw}{r}+V\frac{\partial w}{\partial r}\right)+O(\varepsilon^2)+O(\varepsilon_1)+O(\varepsilon\varepsilon_1) \tag{5}$$

For an inviscid, adiabatic acoustic disturbance, the equation of state gives

$$p=\rho c^2 \tag{6}$$

where c is the mean speed of sound. The expansion will be carried out to $O(\varepsilon)$ in Eqs. (2–6), so that ε_1 must be less than or equal to $O(\varepsilon^2)$ for consistency.

To complete the problem formulation, we need to specify the initial and boundary conditions. The initial conditions will be specified in the next section. We base the boundary conditions on the assumption that the duct walls are lined with point-reacting acoustic materials whose specific acoustic admittances β_1 and β_2 may vary slowly with the axial distance; that is, $\beta_i = \beta_i(x_1)$. For a no-slip mean flow, requiring continuity of particle displacement gives

$$(v-\varepsilon R_j'u)/p = (-1)^j(\beta_j/\rho_0 c)+O(\varepsilon^2) \quad \text{at} \quad r=R_j \tag{7}$$

where R_1 and R_2 are the inner and outer radii of the duct.

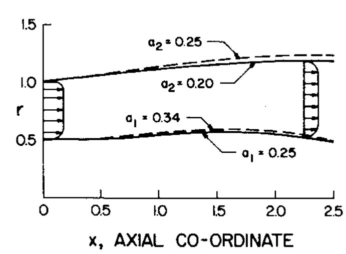

Fig. 2 Approximate model of an annular bypass duct.

Method of Solution

We make lengths, velocities, time and pressure dimensionless using a characteristic length L, the speed of sound c, L/c and $\rho_0 c^2$. Moreover, we use the method of multiple scales[19] to determine an asymptotic solution of Eqs. (2–6), subject to the boundary conditions (7), of the form

$$p(x,r,\theta,t)=\rho_0 c^2 F(x_1,r;\ \varepsilon)\exp(i\phi) \tag{8}$$
$$v(x,r,\theta,t)=cG(x_1,r;\ \varepsilon)\exp(i\phi) \tag{9}$$
$$u(x,r,\theta,t)=cH(x_1,r;\ \varepsilon)\exp(i\phi) \tag{10}$$
$$w(x,r,\theta,t)=cS(x_1,r;\ \varepsilon)\exp(i\phi) \tag{11}$$

where

$$\partial\phi/\partial t=-\omega, \partial\phi/\partial x=k_0(x_1), \partial\phi/\partial\theta=m \tag{12}$$

with constant ω. Here ω is the frequency, m is an integer, the real part of k_0 is the wavenumber, and the imaginary part of k_0 is the attenuation rate, α_0.

The modal description of acoustic waves has been employed in Eqs. (8–12). Hence, for each value of the circumferential mode number m there will exist an infinite number of radial modes corresponding to distinct values of the complex wavenumber k_0. For strong axial variations of the duct, the radial modes interact as they propagate through the duct. However, for the slow variations assumed in this investigation the coupling of the radial modes can be neglected and the various modes studied independently. Thus, Eqs. (8–12) describe a single, arbitrary mode.

In terms of x_1 and ϕ, the derivatives are

$$\partial/\partial t=-\omega(\partial/\partial\phi) \tag{13a}$$
$$\partial/\partial\theta=m(\partial/\partial\phi) \tag{13b}$$
$$\partial/\partial x=k_0(\partial/\partial\phi)+\varepsilon(\partial/\partial x_1) \tag{13c}$$

The functions F, G, H, and S are expanded in the form

$$F=F_0(x_1,r)+\varepsilon F_1(x_1,r)+\dots \tag{14a}$$
$$G=G_0(x_1,r)+\varepsilon G_1(x_1,r)+\dots \tag{14b}$$
$$H=H_0(x_1,r)+\varepsilon H_1(x_1,r)+\dots \tag{14c}$$
$$S=S_0(x_1,r)+\varepsilon S_1(x_1,r)+\dots \tag{14d}$$

Substituting Eqs. (8–14) into Eqs. (2–6), and equating coefficients of like powers of ε, we obtain

Order ε^0:

$$(1/r)(\partial/\partial r)(rG_0)+ikH_0+(im/r)S_0-i\hat{\omega}F_0=0 \tag{15}$$
$$-i\hat{\omega}H_0+(\partial U/\partial r)G_0+ikF_0=0 \tag{16}$$
$$\partial F_0/\partial r-i\hat{\omega}G_0=0 \tag{17}$$
$$-i\hat{\omega}S_0+(im/r)F_0=0 \tag{18}$$
$$G_0+(-1)^{j+1}\beta_j F_0=0 \quad \text{at} \quad r=R_j \tag{19}$$

Order ε:

$$\frac{1}{r}\frac{\partial}{\partial r}(rG_1)+ik_0H_1+\frac{im}{r}S_1-i\hat{\omega}F_1=\zeta_1=-\left(\frac{\partial H_0}{\partial x_1}+U\frac{\partial F_0}{\partial x_1}+V\frac{\partial F_0}{\partial r}\right) \tag{20}$$

$$-i\hat{\omega}H+\frac{\partial U}{\partial r}G_1+ik_0F_1=\zeta_2=-\left[\frac{\partial}{\partial x_1}(UH_0)+\frac{\partial F_0}{\partial x_1}+V\frac{\partial H_0}{\partial r}+F_0\left(U\frac{\partial U}{\partial x_1}+V\frac{\partial U}{\partial r}\right)\right] \tag{21}$$

$$\frac{\partial F_1}{\partial r}-i\hat{\omega}G_1=\zeta_3=-\left(U\frac{\partial G_0}{\partial x_1}+\frac{\partial}{\partial r}(VG_0)\right) \tag{22}$$

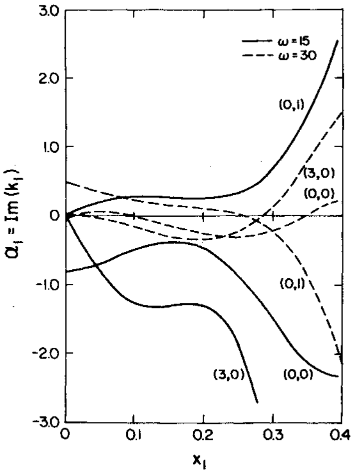

Fig. 3 Correction to the attenuation rate for a duct with $a_1 = 0.25$ and $a_2 = 0.20$ for the (m, n) mode, with m the circumferential and n the radial mode numbers.

$$-i\hat{\omega}S_1 + \frac{im}{r}F_1 = \zeta_4 = -U\left(\frac{\partial S_0}{\partial x_1} + \frac{VS_0}{r} + V\frac{\partial S_0}{\partial r}\right) \tag{23}$$

$$G_1 + (-1)^{j+1}\beta_j F_1 = R_j' H_0 \quad \text{at} \quad r = R_j \tag{24}$$

where

$$\hat{\omega} = \omega - k_0 U \tag{25}$$

Equations (15–19) can be combined into

$$L(F_0) \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{\hat{\omega}^2}\frac{\partial F_0}{\partial r}\right) + \frac{1}{\hat{\omega}^2}\left(\hat{\omega}^2 - k_0{}^2 - \frac{m^2}{r^2}\right)F_0 = 0 \tag{26}$$

$$\partial F_0/\partial r + (-1)^{j+1}i\omega\beta_j F_0 = 0 \quad \text{at} \quad r = R_j \tag{27}$$

The slow scale x_1 appears implicitly in Eqs. (26) and (27). If U, R_j, and β_j are independent of x_1, Eqs. (26) and (27) reduce to those used extensively in the literature to analyze the wave propagation in uniform annular ducts carrying sheared mean flow and lined with materials having uniform acoustic properties (see, for example, Ref. 2). For a given ω, $U(x_1)$, $\beta_j(x_1)$ and $R_j(x_1)$, Eqs. (26) and (27) can be solved to determine the eigenvalue $k_0(x_1)$ and its corresponding eigenfunction $\psi(r;\ x_1)$. The solution can be expressed as

$$F_0 = A(x_1)\psi(r;\ x_1) \tag{28}$$

where $A(x_1)$ is still an undetermined function at this level of approximation. It is determined below at the next level of approximation.

Similarly, Eqs. (20–24) can be combined into

$$L(F_1) = I(F_0, U, V, k_0) \tag{29}$$

$$(\partial F_1/\partial r) + (-1)^{j+1}i\omega\beta_j F_1 = B_j(F_0, U, V, k_0, R_j') \quad \text{at} \quad r = R_j \tag{30}$$

where the functionals I and B_j are defined in the Appendix. Since F_0 and k_0 are solutions of the homogeneous part of Eqs. (29) and (30), the inhomogeneous equations (29) and (30) have a solution if, and only if, a solvability condition is satisfied. This solvability condition yields the desired equation for $A(x_1)$.

To determine the solvability condition, we multiply Eq. (29) by $r\Phi(r;\ x_1)$, where Φ is specified later. Integrating the resulting equation by parts from $r = R_1$ to $r = R_2$ and using the definition of the operator L from Eq. (26) we obtain

$$\int_{R_1}^{R_2} F_1 r L(\Phi)\,dr + \left[\frac{r}{\hat{\omega}^2}\left(\Phi\frac{\partial F_1}{\partial r} - F_1\frac{\partial \Phi}{\partial r}\right)\right]_{R_1}^{R_2} = \int_{R_1}^{R_2} rI\Phi\,dr \tag{31}$$

We choose Φ to be a solution of the so-called adjoint homogeneous problem

$$L(\Phi) = 0 \tag{32}$$

$$\frac{\partial \Phi}{\partial r} - (-1)^j i\omega\beta_j\Phi = 0 \quad \text{at} \quad r = R_j \tag{33}$$

Using Eqs. (32) and (33) and the boundary condition (30), we rewrite Eq. (31) as

$$\omega^2\int_{R_1}^{R_2} rI\Phi\,dr - R_2B_2\Phi(R_2;\ x_1) + R_1B_1\Phi(R_1;\ x_1) = 0 \tag{34}$$

Using the definitions of I and B_j from the Appendix, taking $\Phi = F_0$, and recalling that $F_0 = A(x_1)\psi$, we obtain

$$f(x_1)\frac{dA}{dx_1} + g(x_1)A = 0 \tag{35}$$

where $f(x_1)$ and $g(x_1)$ are quadrature integrals of ψ, U and V and their derivatives. The forms of $f(x_1)$ and $g(x_1)$ and the method of determining dk_0/dx_1 and $\partial\psi/\partial x_1$, on which they depend, are given in the Appendix. The solution of Eq. (35) is

$$A(x_1) = A_0 \exp\left[i\textstyle\int k_1\,dx_1\right] = A_0 \exp\left[i\textstyle\int \varepsilon k_1\,dx\right] \tag{36a}$$

where

$$k_1(x_1) = ig(x_1)/f(x_1) \tag{36b}$$

Thus the form of the dimensional acoustic pressure disturbance is obtained from Eqs. (8, 12, 14a, 28, and 36) as

$$p(x, r, \theta, t) = A_0\rho_0c^2\{\psi(r;\ x_1)\exp\left[i\textstyle\int (k_0 + \varepsilon k_1)\,dx - i\omega t + im\theta\right] + O(\varepsilon)\} \tag{37}$$

where $\psi(r;\ x_1)$ and $k_0(x_1)$ are calculated at each station in the duct *as if* the duct walls and mean flow were parallel, and $k_1(x_1)$ contains the effects of the axial derivatives of the mean flow, ψ and k_0, liner properties and the effect of the slope of the walls. It should be noted that the validity of the solution represented by Eq. (37) does *not* require that $Im(\varepsilon k_1)$ be less than $Im(k_0)$; that is, the correction to the attenuation rate due to streamwise gradients need not be small relative to the zeroth-order (or quasi-parallel) approximation.

Numerical Results

Problem Specifications

A schematic drawing of a high bypass engine is shown in Fig. 1. The method of solution described above has been applied to a simple model of the by-pass duct given by

$$R_2 = 1 + a_2\sin(\pi x/2L)$$

and

$$R_1 = b + (1-b)a_1(x/L)\sin(\pi x/L) \tag{38}$$

where the radius of the outer wall at the entrance to the duct has been chosen as the reference length, b is ratio of the radii of the walls at $x = 0$, L is the dimensionless length of the bypass duct, and a_1 and a_2 are constants that specify the magnitude of the variation in the duct dimensions. For $b = 0.5$ and $L = 2.5$, Fig. 2 shows typical duct configurations described by Eqs. (38). The expansion parameter ε is chosen to be the maximum slope of the inner wall

$$\varepsilon = (dR_1/dx)_{x=L} = (1-b)a_1\pi/L$$

In terms of the slow scale, $x_1 = \varepsilon x$, the duct dimensions are

$$R_2 = 1 + a_2\sin\left[x_1/2a_1(1-b)\right]$$

$$R_1 = b + (x_1/\pi)\sin\left[x_1/a_1(1-b)\right]$$

The entire length of the duct is lined with a thin porous facing sheet backed by cellular cavities of depth d; the specific acoustic admittance of such a liner is described by

$$\beta = 1/\left[R(1 - i\omega/\omega_0) + i\cot\omega d\right]$$

For the numerical calculations discussed below, we take constant liner properties of $R = 2$, $\omega_0 = 100$, and $d = 0.05$.

The axial velocity profile is taken to be uniform within a central core region and to be a quarter sine profile within the boundary layers at the two walls. The boundary layers are assumed to double in thickness over the length of the duct—although this specification is artificial, it serves to demonstrate the applicability of the method. The boundary-layer thickness

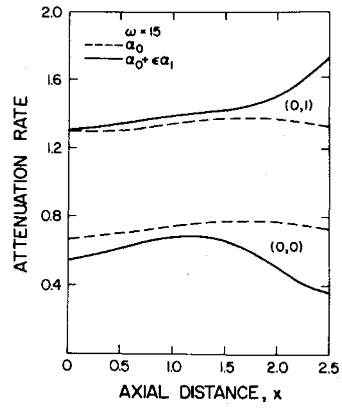

Fig. 4 Comparison of the quasi-parallel attenuation rate, α_0, and the first-order attenuation rate, $\alpha_0+\varepsilon\alpha_1$, for a duct with $a_1 = 0.25$ and $a_2 = 0.20$.

at the duct entrance is $\delta_0(1-b)$, where δ_0 has been taken to be 0.1 unless otherwise noted. The Mach number within the uniform core is taken to be 0.4 at the entrance of the duct and is determined from mass flow considerations at subsequent stations. Previous investigations have shown[18] that the radial velocity component of the mean flow has a negligible effect and hence it is set to zero in the calculations described here.

With the previous problem specifications, a Runge-Kutta scheme is used to solve Eqs. (26) and (27) for $\psi(r;x_1)$ and $k_0(x_1)$. As described in the Appendix, a Runge-Kutta integration is used to obtain $\partial\psi/\partial x_1$ and dk_0/dx_1; $k_1(x_1)$ is evaluated by applying Simpson's rule to the integrals of Eqs. (A7)–(A8).

Discussion of Results

The variation of $Im(k_1)$ with x_1 is shown in Fig. 3 for the first two axially symmetric modes and the third circumferential mode, the dimensionless frequencies 15 and 30, and for the geometrical shape specified by $a_1 = 0.25$ and $a_2 = 0.20$. It can be seen that the effect of the streamwise derivatives is greater at $\omega = 15$ than at $\omega = 30$; this is as expected since the liner is less effective at $\omega = 30$ than it is at $\omega = 15$. At the duct exit, where the inner wall has its maximum slope, most modes exhibit a strong response to the axial variations; however, at the entrance to the duct, where the outer wall has its maximum slope, only two of the modes show a substantial effect of the streamwise derivatives.

Applications of these results to a specific duct whose length, L, is 2.5 (corresponding to $\varepsilon = 0.157$) are shown in Figs. 4 and 5. In both figures a comparison is made between $\alpha_0 = Im(k_0)$, the quasi-parallel approximation, and $\alpha_0+\varepsilon\alpha_1 = Im(k_0+\varepsilon k_1)$, the first-order approximation including the effect of axial derivatives. At $\omega = 15$, the quasi-parallel approximation substantially overpredicts the total attenuation (proportional to the area under the curve) of the fundamental mode and underpredicts the attenuation of the (0, 1) mode. Hence, inclusion of the nonparallel effects is essential for this case. At $\omega = 30$, the total attenuation over the duct length is not considerably affected by the streamwise variations, but it is clear that substantial changes in the local attenuation rate can be achieved by variations in the cross-sectional area.

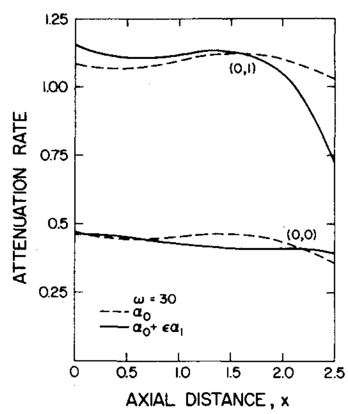

Fig. 5 Comparison of the quasi-parallel attenuation rate, α_0, and the first-order attenuation rate, $\alpha_0+\varepsilon\alpha_1$, for a duct with $a_1 = 0.25$ and $a_2 = 0.20$.

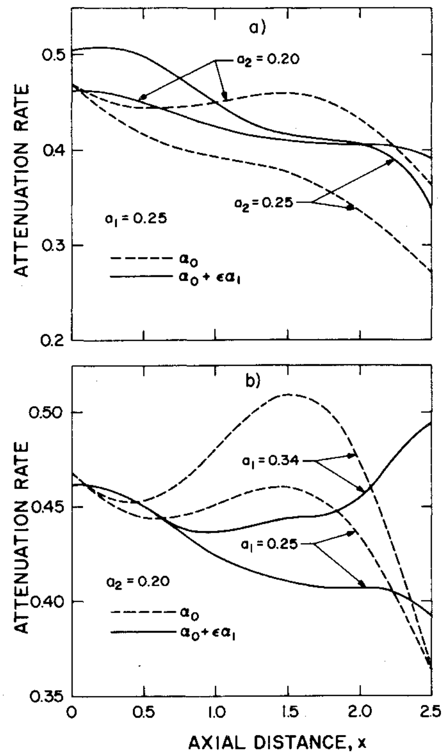

Fig. 6 Comparison of the attenuation of the (0, 0) mode for various wall configurations and $\omega = 30$.

In Fig. 6a a comparison of the effect of two different outer wall variations is made. For the case of $a_2 = 0.25$, the mean flow is expanded to a lower Mach number at the exit than that in the $a_2 = 0.2$ case. The quasi-parallel approximation predicts a lower total attenuation for the $a_2 = 0.25$ case, but inclusion of the nonparallel effects results in a slightly higher attenuation for this case. In Fig. 6b, the results of two different inner wall variations are shown. The entrance and exit Mach numbers are the same for both configurations, but the $a_1 = 0.34$ case has a slightly higher Mach number within the duct than does the $a_1 = 0.25$ case. The quasi-parallel approximation indicates that the additional constriction of the duct provided by $a_1 = 0.34$ results in a slightly higher attenuation. Inclusion of the effects of the axial derivatives does not change this general conclusion but does result in a lower attenuation level and shifts the point of peak attenuation away from the point of maximum deflection of the inner wall.

Figure 7 compares the attenuation levels corresponding to two different mean boundary-layer thicknesses. As is to be expected on the basis of numerous parallel-flow studies, the zeroth-order approximation predicts a lower attenuation when the boundary layer is thinner. However, it can also be seen that the variable-area effects are much larger for the thinner boundary layer; thus, for the specific case examined here, the effect of the thicker boundary-layer is much less than is expected from a parallel-flow analysis. Since the variable-area effects appear to be sensitive to a proper modeling of the boundary-layer growth, it may be necessary to couple such growth to the streamwise velocity gradients. Future investigations will examine this in

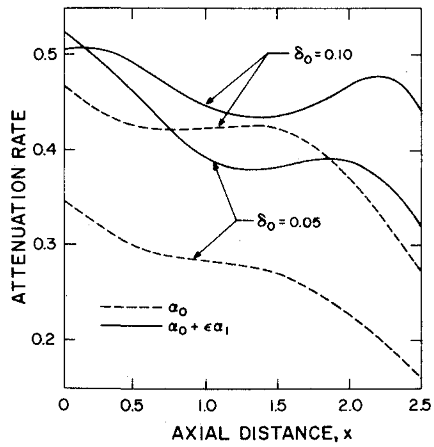

Fig. 7 Effect of the mean boundary-layer thickness on the attenuation of the (0, 0) mode for $\omega = 30$, $a_1 = 0.34$, $a_2 = 0.25$, $\varepsilon = 0.214$.

more detail as well as examine the effects of the radial mean-flow velocity and axial variations of the liner properties.

Summary

The method of multiple scales is used to obtain a first-order, uniformly-valid asymptotic expansion for the transmission and attenuation of acoustic waves in an acoustically-lined annular duct with a varying cross section and carrying sheared mean flow. The analysis also takes into account the growth of the boundary layer as well as any slow variations in the liner properties.

To first order, the dimensionless acoustic pressure has the form

$$p = A_0\psi(r, x_1)\exp\{i\textstyle\int[k_0(x_1)+\varepsilon k_1(x_1)]\,dx - i\omega t + im\theta\}$$

where A_0 is a normalizing constant, $x_1 = \varepsilon x$ is a slow scale characterizing the distortion of the wave due to the streamwise variations, ε is the maximum slope of the wall, ω is the dimensionless frequency, m is an integer, and $k_0+\varepsilon k_1$ is the dimensionless propagation constant. Here, $\psi(r, x_1)$ and k_0 are, respectively, the eigenfunction and eigenvalue for the quasi-parallel problem (at each axial location, the duct is replaced by an infinite one having a uniform cross section equal to the local value). According to this model, the attenuation rate at any location is independent of the attenuation rates at all other locations. The perturbed part εk_1 is due to the coupling of the attenuation rate at a given location with the attenuation rate at the preceding location; it is a function of the axial derivatives of ψ, U, and k_0.

The numerical results presented in the paper show that significant errors may be introduced by replacing the variable duct by a uniform duct or by using a quasi-parallel approximation. More numerical results are needed to ascertain the importance of all the liner, geometry, and flow parameters on the attenuation of sound in an annular duct with a variable cross section.

Appendix

Inhomogeneous terms of Eqs. (29) and (30)

$$I(F_0, U, V, k_0) = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r\zeta_3}{\hat\omega^2}\right) + \frac{ik_0}{\hat\omega^2}\zeta_2 + \frac{im}{r\hat\omega^2}\zeta_4 + \frac{i}{\hat\omega}\zeta_1 \tag{A1}$$

$$B_j(F_0, U, V, k_0, R_j') = \left\{\zeta_3 + iR_j'\left(k_0F_0 - \frac{\partial U}{\partial r}\frac{\partial F_0}{\partial r}/\hat\omega\right)\right]_{r=R_j} \tag{A2}$$

Introduce the following notation

$$g_1 = r\left[\frac{\partial}{\partial x_1}(k_0/\hat\omega)\right]\Big/\hat\omega$$

$$g_1{}^r = r\left[V - \frac{\partial}{\partial x_1}\left(\frac{\partial U}{\partial r}\Big/\hat\omega^2\right)\right]\Big/\hat\omega$$

$$g_1{}^x = r(U + k_0/\hat\omega)/\hat\omega$$

$$g_1{}^{rx} = -r\frac{\partial U}{\partial r}\Big/\hat\omega^3$$

$$g_2 = k_0r\left[\frac{\partial}{\partial x_1}(k_0U/\hat\omega) + V\frac{\partial U}{\partial r}(1 + k_0{}^2/\hat\omega^2) + U\frac{\partial U}{\partial x_1}\right]\Big/\hat\omega^2$$

$$g_2{}^x = k_0r(1 + k_0U/\hat\omega)/\hat\omega^2$$

$$g_2{}^r = k_0r\left[Vk_0/\hat\omega - \frac{\partial}{\partial x_1}\left(U\frac{\partial U}{\partial r}\Big/\hat\omega^2\right) - V\frac{\partial}{\partial r}\left(\frac{\partial U}{\partial r}\Big/\hat\omega^2\right)\right]\Big/\hat\omega^2$$

$$g_2{}^{rx} = -k_0rU\frac{\partial U}{\partial r}\Big/\hat\omega^4$$

$$g_2{}^{rr} = -k_0rV\frac{\partial U}{\partial r}\Big/\hat\omega^4$$

$$g_3{}^r = -\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(V/\hat\omega)/\hat\omega^2 + rU\frac{\partial}{\partial x_1}(1/\hat\omega)/\hat\omega^2\right]$$

$$g_3{}^{rr} = -r\left[\frac{\partial}{\partial r}(V/\hat\omega)/\hat\omega^2 + U\frac{\partial}{\partial x_1}(1/\hat\omega)/\hat\omega^2\right] - \frac{\partial}{\partial r}(Vr/\hat\omega^3)$$

$$g_3{}^{rrr} = -rV/\hat\omega^3$$

$$g_3{}^{rrx} = -rU/\hat\omega^3$$

$$g_3{}^{rx} = -\frac{\partial}{\partial r}(Ur/\hat\omega^3)$$

$$g_4 = m^2\left[U\frac{\partial}{\partial x_1}(1/\hat\omega) + V\frac{\partial}{\partial r}(1/\hat\omega)\right]r\hat\omega^2$$

$$g_4{}^r = m^2V/r\hat\omega^3$$

$$g_4{}^x = m^2U/r\hat\omega^3 \tag{A3}$$

and

$$N = g_1 + g_2 + g_4$$

$$N_r = g_1{}^r + g_2{}^r + g_3{}^r + g_4{}^r$$

$$N_{rr} = g_2{}^{rr} + g_3{}^{rr}$$

$$N_{rrr} = g_3{}^{rrr}$$

$$N_x = g_1{}^x + g_2{}^x + g_4{}^x$$

$$N_{rx} = g_1{}^{rx} + g_2{}^{rx} + g_3{}^{rx}$$

$$N_{rrx} = g_3{}^{rrx} \tag{A4}$$

With the above equations, $I(F_0, U, V, k_0)$ becomes

$$\begin{aligned} I = -iA\{&N\Psi + N_r\partial\Psi/\partial r + N_x\partial\Psi/\partial x_1 + N_{rx}\partial^2\Psi/\partial x_1\partial r + \\ &N_{rr}\partial^2\Psi/\partial r^2 + N_{rrr}\partial^3\Psi/\partial r^3 + N_{rrx}\partial^3\Psi/\partial r^2\partial x_1\}/r - \\ &i\,\partial A/\partial x_1\{N_x\Psi + N_{rx}\partial\Psi/\partial r + N_{rrx}\partial^2\Psi/\partial r^2\}/r \end{aligned} \tag{A5}$$

and B_j can be written as

$$B_j = iA\{k_0R_j'\psi + [\partial V/\partial r - R_j'\,\partial U/\partial r](\partial\psi/\partial r)/\omega\} \tag{A6}$$

Coefficients of Eq. (35)

The functions $f(x_1)$ and $g(x_1)$ of Eq. (35) are calculated by performing the following integrations across the duct:

$$f(x_1) = \int_{R_1}^{R_2}[N_x\Psi + N_{rx}\partial\Psi/\partial r + N_{rrx}\partial^2\Psi/\partial r^2]\Psi\,dr \tag{A7}$$

and

$$\begin{aligned} g(x_1) = &\left(k_0R_j'R_j\Psi^2/\omega^2 + R_j(\partial V/\partial r - R_j'\,\partial U/\partial r)\Psi(\partial\Psi/\partial r)/\omega^3\right)_{j=1}^{j=2} \\ &+ \int_{R_1}^{R_2}[N\Psi + N_r\partial\Psi/\partial r + N_{rr}\partial^2\Psi/\partial r^2 + \\ &N_{rrr}\partial^3\Psi/\partial r^3 + N_x\partial\Psi/\partial x_1 + N_{rx}\partial^2\Psi/\partial r\,\partial x_1 + \\ &N_{rrx}\partial^3\Psi/\partial r^2\,\partial x_1]\Psi\,dr \end{aligned} \tag{A8}$$

Determination of $\partial\Psi/\partial x_1$ and dk_0/dx_1

In order to evaluate the integrals of Eqs. (A7) and (A8), it is necessary to know dk_0/dx_1, $\partial\Psi/\partial x_1$, $\partial^2\Psi/\partial r\,\partial x_1$, and $\partial^3\Psi/\partial r^2\,\partial x_1$. In a previous investigation[18] for plane ducts, a finite-difference approximation has been used to obtain dk_0/dx_1 after solving for k_0 at two adjacent stations, and then $\partial\Psi/\partial x_1$, etc., have been eliminated from Eq. (A8) by use of Liebnitz's rule. This latter manipulation becomes very tedious in the case of an axisymmetric duct, and thus an alternate approach has been developed that has the additional advantage of not requiring a finite-difference approximation for dk_0/dx_1.

Differentiation of Eqs. (26) and (27) with respect to x_1 yields

$$L(\partial\Psi/\partial x_1) = h_2\, dk_0/dx_1 + h_1 \qquad \text{(A9)}$$

where

$$h_2(r;\ x_1) = 2(\hat{\omega}U + k_0)\Psi/\hat{\omega}^2 - 2(\hat{\omega} + Uk_0)\,\partial U/\partial r\, \partial\Psi/\partial r/\hat{\omega}^4$$

$$h_1(r;\ x_1) = 2k_0\, \partial U/\partial x_1\, \Psi/\hat{\omega} - 2k_0(\hat{\omega}\, \partial^2 U/\partial x_1\, \partial r + k_0\, \partial U/\partial r\, \partial U/\partial x_1)\, \partial\Psi/\partial r/\hat{\omega}^4$$

and

$$\frac{\partial}{\partial r}(\partial\Psi/\partial x_1) + (-1)^{j+1} i\omega\beta_j \partial\Psi/\partial x_1 = R_j'[-\partial^2\Psi/\partial r^2 + (-1)^j i\beta_j\omega\, \partial\Psi/\partial r] + (-1)^j i\omega\, d\beta/dx_1\, \Psi \quad \text{on} \quad r = R_j \qquad \text{(A10)}$$

Equations (A9) and (A10) will have a solution for $\partial\Psi/\partial x_1$ only if the inhomogeneous term is orthogonal to Ψ, which satisfies the corresponding homogeneous equations. Application of this integrability condition yields

$$(dk_0/dx_1)\int_{R_1}^{R_2} h_2 r\Psi\, dr = -\int_{R_1}^{R_2} h_1 r\Psi\, dr + \Big(\Psi R_j[(-1)^j R_j' i\beta_j\omega\, \partial\Psi/\partial r + (-1)^j i\omega\Psi\, d\beta_j/dx_1 - R_j'\, \partial^2\Psi/\partial r^2]/\omega^2\Big)_{R_1}^{R_2} \qquad \text{(A11)}$$

The coefficients of Eq. (A11) can be evaluated numerically using Simpson's rule, and the value of dk_0/dx_1 is thus determined. With dk_0/dx_1 known, Eq. (A9) can be integrated by substituting

$$\partial\Psi/\partial x_1 = \Psi(r;\ x_1)E(r;\ x_1)$$

and obtaining $E(r;\ x_1)$ from integrals of $h_2\, dk_0/dx_1 + h_1$ and Ψ. To complete the specification, the value of $\partial\Psi/\partial x_1$ at the inner wall is obtained from the fact that $\Psi(r;\ x_1)$ is normalized to a constant value at the inner wall. Thus, $d\Psi = 0$, or

$$\partial\Psi/\partial x_1 = -R_1'\, \partial\Psi/\partial r \quad \text{on} \quad r = R_1 \qquad \text{(A12)}$$

Alternatively, Eq. (A9) can be integrated using a Runge-Kutta scheme for two independent values of dk_0/dx_1, with Eqs. (A10) and (A12) providing initial conditions at $r = R_1$. A linear superposition of the two solutions to satisfy the boundary condition (A10) at $r = R_2$ then yields the value of dk_0/dx_1 and $\partial\Psi/\partial x_1$ (as well as $\partial^2\Psi/\partial r\, \partial x_1$ and $\partial^3\Psi/\partial r^2\, \partial x_1$).

Both numerical processes described above have been tested and found to yield consistent results. The numerical results presented in this paper have been generated with the second technique.

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